Pricing path-dependent options using optimized functional quantization

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What is (quadratic) Functional Quantization ?

$\triangleright$ $X : \Omega \longrightarrow H$, $(H, (\,.\,|\,.\,))$ separable Hilbert space

$$\mathbb{E}|X|^2 < +\infty.$$

$\triangleright$ When $H = \mathbb{R}$, $\mathbb{R}^d$ $\equiv$ Vector Quantization of a random vector X.

$\triangleright$ When $H = L^2([0,T], dt) =: L^2_T$ $\equiv$ Functional Quantization of a process $X = (X_t)_{t\in[0,T]}$.

$$\boxed{\text{Discretization of the path space } H = L^2([0,T], dt)}$$

using

▷ N-quantizer (or N-codebook or N-quantization grid) :

$$\alpha := \{\alpha_1, \ldots, \alpha_N\} \subset L^2_T.$$

▷ Discretization by α-quantization

$$X \rightsquigarrow \widehat{X}^{\alpha} : \Omega \to \alpha := \{\alpha_1, \ldots, \alpha_N\}.$$

$$\widehat{X}^{\alpha} := \mathrm{Proj}_{\alpha}(X)$$

where

Proj_{α} denotes the projection on α following the nearest neigbour rule.

QUESTION : **What do we know about $X - \widehat{X}^{\alpha}$ and $\widehat{X}^{\alpha}$?**

$\triangleright$ Pointwise induced erreur : for every $\omega \in \Omega$

$$|X(\omega) - \widehat{X}^{\alpha}(\omega)|_{H} = \mathrm{dist}_{H}(X(\omega), \Gamma_{x}) = \min_{1 \le i \le N} |X(\omega) - \alpha_i|_{H}$$

$\triangleright$ Mean induced error (or quantization quadratic error) :

$$\|X - \widehat{X}^{\alpha}\|_2^2 = \mathbb{E}\left(\min_{1 \le i \le N} |X - \alpha_i|_H^2\right)$$

$\triangleright$ Distribution of $\widehat{X}^{x}$: weights associated to each x_i

$$\mathbb{P}(\widehat{X}^{\alpha} = \alpha_i) = \mathbb{P}(X \in C_i(\alpha)), \qquad i = 1, \ldots, N$$

where $C_i(\alpha)$ denotes the Voronoi cell of x_i (w.r.t. α) defined by

$$C_i(\alpha) := \left\{ \xi \in H : |\xi - \alpha_i|_H = \min_{1 \le j \le N} |\xi - \alpha_j|_H \right\}.$$

Optimal (Quadratic) Quantization

The distorsion (square of the quantization error)

$$D_N^X \quad : \quad H^N \longrightarrow \mathbb{R}_+$$

$$\alpha = (\alpha_1, \ldots, \alpha_N) \longmapsto \|X - \widehat{X}^{\alpha}\|_2^2 = \mathbb{E}\left(\min_{1 \le i \le N} |X - \alpha_i|_H^2\right)$$

– Lipschitz continuous for the norm-topolgy

– l.s.c for the weak topology

One shows *by induction on* N that

D_N^X reaches a minimum at an (optimal) quantizer of full size N

If $N = 1$, the optimal 1-quantizer is $(\mathbb{E}\, X)$ and

the induced error is $\sigma(|X|_H)$.

Stationary Quantizers

▷ The distortion D_N^X is differentiable at the full size N-codebooks α and

$$\nabla D_N^X(\alpha) = 2\left(\int_{C_i(\alpha)} (\alpha_i - \xi)\mathbb{P}_X(d\xi)\right)_{1\le i\le N} = 2\left(\mathbb{E}(\widehat{X}^{\alpha} - X)\mathbf{1}_{\{\widehat{X}^{\alpha}=x_i\}}\right)_{1\le i\le N}$$

▷ Definition : If $\alpha \subset H^N$ is a zero of $\nabla D_N^X(\alpha)$, then α is a stationary quantizer .

$$\boxed{\nabla D_N^X(\alpha) = 0 \quad \Longleftrightarrow \quad \widehat{X}^{\alpha} = \mathbb{E}\left(X \,|\, \widehat{X}^{\alpha}\right)}$$

since

$$\sigma(\widehat{X}^{\alpha}) = \sigma(\{X \in C_i(\alpha)\}, i = 1, \ldots, N)$$

▷ An optimal quantizer α is stationary (hence first moment of X and $\widehat{X}$ coincide).

Quantization rate in $H = \mathbb{R}^d$

$\triangleright$ THEOREM (Zador and al., of 1963 à 2000) Let $X \in L^{2+}(\mathbb{P})$ and $\mathbb{P}_X(d\xi) = \varphi(\xi)\, d\xi$. If $(\alpha^{N,*})_{N\geq 1}$ is optimal then

$$\boxed{\|X - \widehat{X}^{\alpha^{N,*}}\|_2 \sim \widetilde{J}_{2,d} \times \left(\int_{\mathbb{R}^d} \varphi^{\frac{d}{d+2}}(u)\, du\right)^{\frac{1}{d}+\frac{1}{2}} \times \frac{1}{N^{\frac{1}{d}}} \quad \text{as} \quad N \to +\infty.}$$

The true value of $\widetilde{J}_{2,d}$ is unknown for $d \geq 3$ but

$$\widetilde{J}_{2,d} \sim \sqrt{\frac{d}{2\pi e}} \approx \sqrt{\frac{d}{17,08}} \quad \text{as} \quad d \to +\infty.$$

CONCLUSION : For every N the same rate as with "naive" product-grids for the $U([0,1]^d)$ disribution with $N = m^d$ points + the best constant

The 1-dimension ...

▷ THEOREM $H = \mathbb{R}$. If $\mathbb{P}_X(d\xi) = \varphi(\xi)\, d\xi$ with $\log f$ concave, then

$$\forall\, N \geq 1, \qquad \text{argmin} D_N^X = \{x^{(N)}\}$$

EXAMPLES : The normal distribution, the gamma distributions, etc.

▷ Voronoi cells : $C_i(x) = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[,\;\; x_{i+\frac{1}{2}} = \frac{x_{i+1}+x_i}{2}$.

▷ $\nabla D_N^X(x) = 2\left(\displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (x_i - \xi)\varphi(\xi)d\xi\right)_{1\leq i\leq N}$ and the Hessian

$$\nabla^2 D_N^X(x) = \ldots \text{ only uses } \varphi \text{ and } \int_0^x \xi\varphi(\xi)d\xi)$$

▷ If $X \sim \mathcal{N}(0;1)$: only $\text{erf}(x)$ and $e^{-\frac{x^2}{2}}$ are involved.

▷ Instant search for the unique zero using a Newton-Raphson descent on $\mathbb{R}^N$ ... with an arbitrary accuracy.

$$\boxed{x(t+1) = x(t) - \left(\nabla^2 D_N^X(x(t))\right)^{-1} (\nabla D_N^X(x(t)))}$$

▷ For $\mathcal{N}(0;1)$ and every $N = 1, \ldots, 400$, tabulation within 10^{-14} accuracy of

$$x^{(N)} = (x_1^{(N)}, \ldots, x_N^{(N)}) \quad \text{and} \quad \mathbb{P}(X \in C_i(x^{(N)})),\ i = 1, \ldots N.$$

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Functional Quantization (of the Brownian motion)

▷ $H = L^2_T := L^2([0,T]), dt)$, $(f|g) = \int_0^T f(t)g(t)dt$, $|f|_{L^2_T} = \sqrt{(f|f)}$.

▷ The Brownian motion W : centered Gaussian process with covariance operator : $C_B(f) : f \longmapsto (t \mapsto \int_{[0,T]^2} (s \wedge t) f(s) ds)$

▷ Diagonalization of $C_B \Longrightarrow$ Karhunen-Loève system ($\equiv$ CPA of W)

$$\boxed{e_n^W(t) = \sqrt{2T}\sin\left((n-\frac{1}{2})\pi\frac{t}{T}\right), \qquad \lambda_n = \left(\frac{T}{\pi(n-\frac{1}{2})}\right)^2, \; n \geq 1}$$

$$\begin{aligned} W_t &\overset{L^2_T}{=} \sum_{n\geq 1} (W|e_n)_2 e_n^W(t) = \sum_{n\geq 1} \sqrt{\lambda_n}\,\xi_n e_n^W(t) \\ \xi_n &\sim \mathcal{N}(0;1), \quad n \geq 1, \quad \text{iid}. \end{aligned}$$

$\triangleright$ THEOREM (Luschgy-P., JFA (2000) and AP (2003))

$\alpha^N = (\alpha_1^N, \cdots, \alpha_N^N)$ sequence of optimal N-quantizers.

$\triangleright$ $\alpha^N \subset \text{span}\{e_1^W, \ldots, e_{d(N)}^W\}$ with $d(N) \sim \log(N)$.

$\triangleright$ $\|W - \widehat{W}^{\alpha^N}\|_2 \sim \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{\log(N)}}$ as $N \to \infty$. $\quad (\frac{\sqrt{2}}{\pi} = 0.2026\ldots)$

$\triangleright$ Pythagore dimension reduction

$$(\mathcal{O}_N) \left\{ \begin{array}{l} \|W - \widehat{W}^{\alpha^N}\|_2^2 = \min_{\beta \subset \mathbb{R}^{d(N)}, |\beta| = N} \|Z - \widehat{Z}^{\beta}\|_2^2 + \sum_{k \geq d(N)+1} \lambda_k \\ Z \sim \bigotimes_{k=1}^{d(N)} \mathcal{N}(0, \lambda_k). \end{array} \right.$$

Then

$$\widehat{W}^{\alpha^N} = \sum_{k=1}^{d(N)} (\widehat{Z}^{\beta^*(N)})_k e_k^W.$$

Functional Quantization : numerical aspects $(T = 1)$

▷ Good news : $(\mathcal{O}_N)$ is a finite dimensional quantization optimization problem.

▷ Bad news : $\lambda_1 = 0.40528...$ and $\lambda_2 = 0.04503... \approx \lambda_1/10\,!\,!\,!$

▷ A way out :

$$(\mathcal{O}_N) \Leftrightarrow \begin{cases} N\text{-optimal quantization of } \displaystyle\bigotimes_{k=1}^{d(N)} \mathcal{N}(0,1) \\ \text{for the norm } |(z_1,\dots,z_{d(N)})|^2 = \sum_{k=1}^{d(N)} \lambda_k z_k^2. \end{cases}$$

▷ A toolbox : (variants of) *Competitive Learning Vector Quantization* and multi-dim fixed point *"Lloyd I procedure"* (see G.P.-J.P., MCMA, 2003), etc (in progress) :

$$\widehat{Z}^{(\alpha^N)(n+1)} = \mathbb{E}(Z \,|\, \widehat{Z}^{(\alpha^N)(n)}), \qquad (\alpha^N)(0) \subset \mathbb{R}^d.$$

▷ As a result :

• Optimized stationary codebooks $\beta^*(N)$ for the Brownian Motion

$N = 1$ up to $10\,000$ with $d(N) = 1$ up to 9.

• Computation of the companion parameters :

– Weights = distribution of $\widehat{W}^{\alpha^N}$, $N \geq 1$, and

– quantization errors $\|W - \widehat{W}^{\alpha^N}\|_2$, $N \geq 1$.

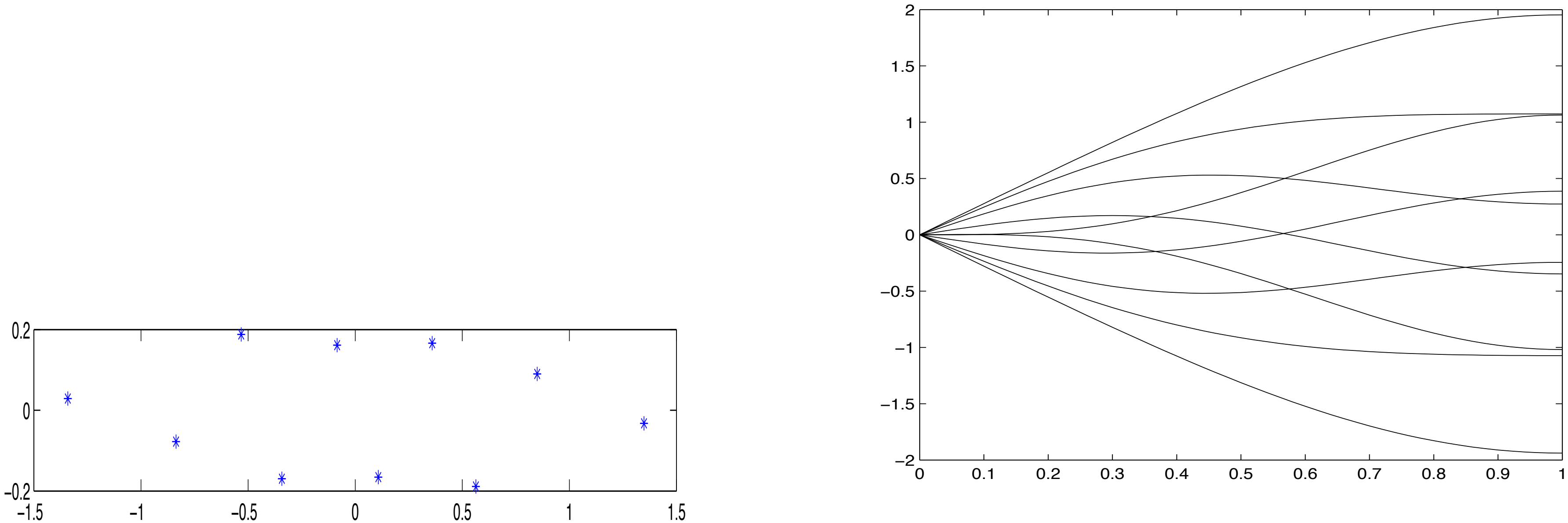

Fig. 1: Optimized FQ of the Brownian motion $N = 10$: the point in $\mathbb{R}^2$ *vs* the paths in the K-L basis

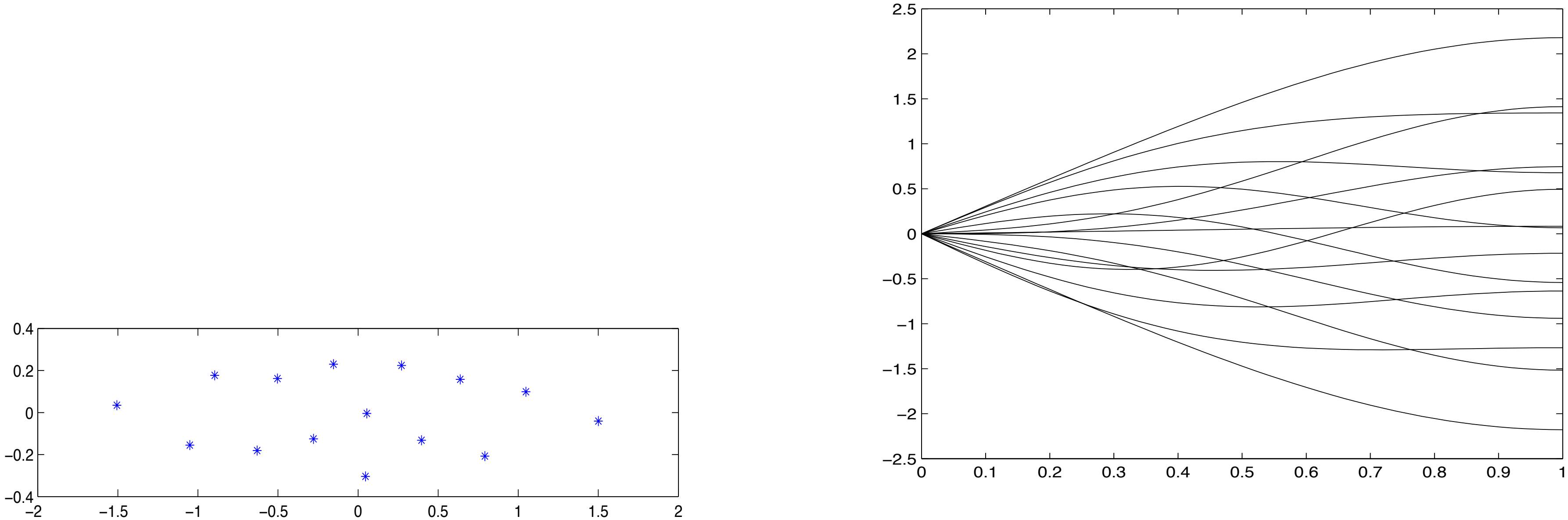

Fig. 2: Optimized FQ of the Brownian motion $N = 15$: the point in $\mathbb{R}^2$ *vs* the paths in the K-L basis

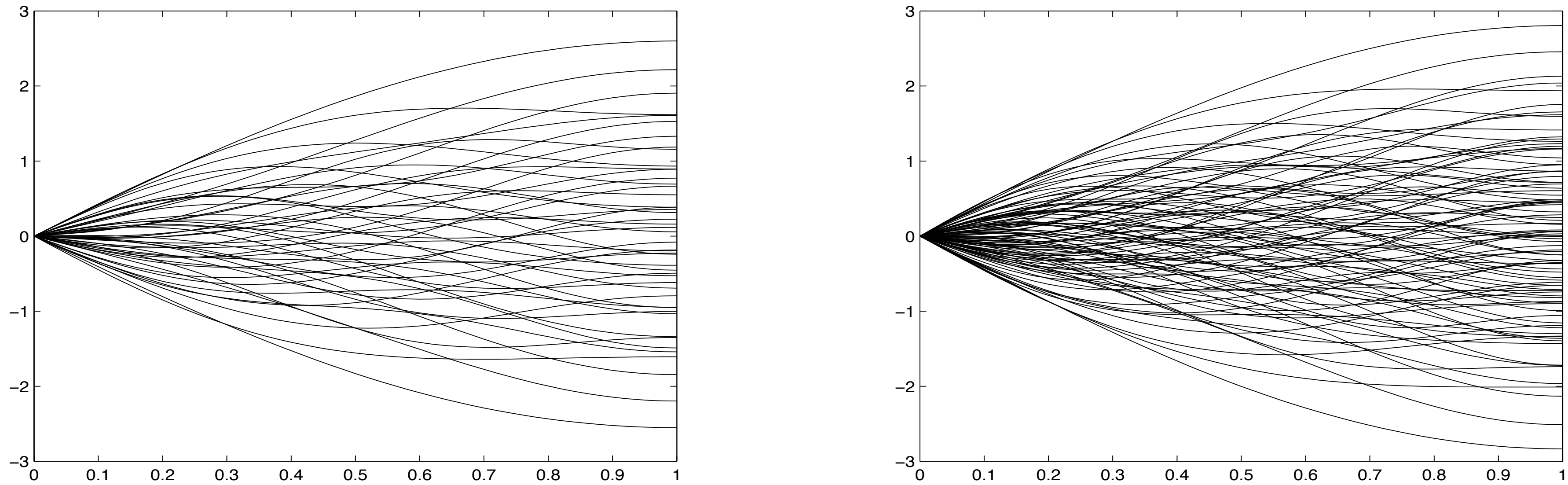

FIG. 3: Optimized Quantization of the Brownian motion $N = 48$ and $N = 96$

Rate optimal quantization of diffusions

▷ $dX_t = b(t, X_t)dt + \vartheta(t, X_t)dW_t \qquad b, \vartheta$ continuous with linear growth.

▷ α^N, $N \geq 1$, sequence of rate optimal N-quantizers of W.

▷ $dx_i^{(N)}(t) = (b(t, x_i^{(N)}(t)) - \frac{1}{2}\vartheta\vartheta'(t, x_i^{(N)}(t)))dt + \vartheta(t, x_i^{(N)}(t))d\alpha_i^N(t)$

▷ Theorem Luschgy-P., SPA (2006) The sequence $(x^{(N)})_{N\geq 1}$ is rate optimal

$$\| \, |X - \widetilde{X}^{x^{(N)}}|_{L_T^2} \|_2 = O\left(\frac{1}{(\log(N))^{\frac{1}{2}}}\right) \qquad (\asymp \text{ if } \vartheta \geq \varepsilon_0 > 0)$$

where $\widetilde{X}^{x^{(N)}}$ is a *non-Voronoi* but explicit quantizer defined

$$\widetilde{X}_t^{x^{(N)}} = \sum_{k=1}^{N} x_i^{(N)}(t)\mathbf{1}_{\{\widehat{W}^{\alpha^N} = \alpha_i^N\}}$$

Numerical Integration (I-II) : quadrature formulae

Let $F : L^2[0,T],dt) \longrightarrow \mathbb{R}$ be a functional and let $\alpha \in L^2([0,T],dt)$ be an N-quantizer.

$$\mathbb{E}\,F(\widehat{W}^{\alpha}) = \sum_{i=1}^{N} F(\alpha_i)\mathbb{P}(W \in C_i(\alpha))$$

▷ If F is Lipshitz.

$$\boxed{\left|\mathbb{E}F(W) - \mathbb{E}F(\widehat{W}^{\alpha})\right| \le [F]_{\rm Lip}\mathbb{E}\left|W - \widehat{W}^{\alpha}\right| \le [F]_{\rm Lip}\|W - \widehat{W}^{\alpha}\|_2}$$

$\triangleright$ If F is $\mathcal{C}^1$ and DF is Lipschitz and α a stationary quantizer .

$$F(W) = F(\widehat{W}^{\alpha}) + DF(\widehat{W}^{\alpha}).(W - \widehat{W}^{\alpha}) + (DF(\widehat{W}^{\alpha}) - DF(\zeta)).(W - \widehat{W}^{\alpha})$$

$\zeta \in (W, \widehat{W}^{\alpha})$, hence

$$|\mathbb{E}F(W) - \mathbb{E}F(\widehat{W}^{\alpha}) - \underbrace{\mathbb{E}\left(DF(\widehat{W}^{\alpha}).(W - \widehat{W}^{\alpha})\right)}_{=0}| \quad \leq \quad [DF]_{\rm Lip}\mathbb{E}\left|W - \widehat{W}^{\alpha}\right|^2$$

so that

$$\boxed{\left|\mathbb{E}F(W) - \mathbb{E}F(\widehat{W}^{\alpha})\right| \leq [DF]_{\rm Lip}\|X - \widehat{W}^{\alpha}\|_2^2}$$

since

$$\mathbb{E}\left(DF(\widehat{W}^{\alpha}).(W - \widehat{W}^{\alpha})\right) = \mathbb{E}\left(DF(\widehat{W}^{\alpha}).\mathbb{E}(W - \widehat{W}^{\alpha}\,|\,\widehat{W}^{\alpha})\right) = 0.$$

Typical functionals

– Fonctionnals $|\,.\,|_{L^2_T}$-continuous at every $\omega \in \mathcal{C}([0,T])$?

$$F(\omega) := \int_0^T f(t,\omega(t))dt$$

where$f$ is *locally Lipschitz continuous*, namely

$$|f(t,u) - f(t,v)| \leq C_f |u-v|(1 + g(t,u) + g(t,v)).$$

Example : The Asian payoff in B-S model

$$F(\omega) = \exp(-rT)\left(\frac{1}{T}\int_0^T \exp(\sigma\omega(t) + (r-\sigma^2/2)t)dt - K\right)_+ .$$

Numerical Integration (III) : log-Romberg

▷ $F : L^2_T \longrightarrow \mathbb{R}$, 3 times $|\,.\,|_{L^2_T}$-differentiable with bounded differential.

▷ $\widehat{W}^{(N)}$, $N \geq 1$, stationary rate-optimal quantizations

▷
$$\begin{aligned} F(W) &= F(\widehat{W}^{(N)}) + DF(\widehat{W}^{(N)}).(W - \widehat{W}^{(N)}) \\ &\quad + \frac{1}{2} D^2 F(\widehat{W}^{(N)}).(W - \widehat{W}^{(N)})^{\otimes 2} + \frac{1}{6} D^3 F(\widehat{W}^{(N)}).(W - \widehat{W}^{(N)})^{\otimes 3}. \end{aligned}$$

$$\boxed{\mathbb{E}F(W) = \mathbb{E}F(\widehat{W}^{(N)}) + \frac{1}{2}\mathbb{E}\left(D^2 F(\widehat{W}^{(N)}).(W - \widehat{W}^{(N)})^{\otimes 2}\right) + o\left((\log N)^{-\frac{3}{2}+\varepsilon}\right).}$$

CONJECTURE : $$\mathbb{E}\left(D^2F(\widehat{W}^{(N)}).(W-\widehat{W}^{(N)})^{\otimes 2}\right) \sim \frac{c}{\log N}, \qquad N \to \infty$$

Set

$$M \ll N \quad (e.g.\ M \approx N/4)$$

and $\forall\, \varepsilon > 0$

$$\mathbb{E}(F(W)) = \frac{\log N \times \mathbb{E}(F(\widehat{W}^{(N)})) - \log M \times \mathbb{E}(F(\widehat{W}^{(M)}))}{\log N - \log M} + o\left((\log N)^{-\frac{3}{2}+\varepsilon}\right),$$

Possible variant (mainly for *product quantizations*, B.Wilbertz (Trier, 2005)) :

Replace $\log(N)$ by $1/\|W - \widehat{W}^{(N)}\|_2^2$.

Application : Asian option in a Heston stochastic volatility model

▷ The dynamics : Let ϑ, k, a s.t. $\vartheta^2/(4ak) < 1$.

$$\begin{aligned} dS_t &= S_t(r\,dt + \sqrt{v_t})dW_t^1, \qquad S_0 = s_0 > 0, \quad \text{(risky asset)} \\ dv_t &= k(a - v_t)dt + \vartheta\sqrt{v_t}dW_t^2,\ v_0 > 0 \quad \text{with } <W^1, W^2>_t = \rho\, t,\ \rho \in [-1,1]. \end{aligned}$$

▷ The payoff and the premium :

$$\text{AsCall}^{Hest} = e^{-rT}\mathbb{E}\left(\left(\frac{1}{T}\int_0^T S_s ds - K\right)_+\right).$$

(no closed form available)

▷ The procedure : • Projection of W^1 on W^2

$$S_t = s_0 \exp\left((r - \frac{1}{2}\bar{v}_t)t + \rho\int_0^t \sqrt{v_s}dW_s^2\right)\exp\left(\sqrt{1-\rho^2}\int_0^t \sqrt{v_s}d\widetilde{W}_s^1\right)$$

- Chaining rule for conditional expectations

$$\mathrm{AsCall}^{Hest}(s_0,K)=e^{-rT}\mathbb{E}\left(\mathbb{E}\left(\left(\frac{1}{T}\int_0^T S_s ds-K\right)_+ |\sigma((v_t)_{0\le t\le T})\right)\right)$$

- solving the quantization ODE's for (v_t) (by a Runge-Kuta scheme)

$$dy_i(t)=\left(k(a-y_i(t)-\frac{\vartheta^2}{4k}\right)dt+\vartheta\sqrt{y_i(t)}d\alpha_i^N(t),\ i=1,\ldots,N.$$

Set the (non-Voronoi but rate optimal) N-quantization of (v_t,S_t) by

$$\widetilde{v}_t^{n,N}=\sum_{\underline{i}} y_{\underline{i}}^{n,N}(t)\mathbf{1}_{C_{\underline{i}}(\chi^N)}(W^2).$$

and $$\widetilde{S}_t^{n,N} = \sum_{1\le i,j\le N} s_{i,j}^{n,N}(t)\mathbf{1}_{\alpha_i^N}(\widetilde{W}^1)\mathbf{1}_{\alpha_j^N}(W^2).$$

with

$$\begin{aligned} s_{i,j}^{n,N}(t) &= s_0 \exp\left(t\left((r-\frac{\rho a k}{\vartheta}) + \bar{y}_j^{n,N}(t)(\frac{\rho k}{\vartheta}-\frac{1}{2})\right) + \frac{\rho}{\vartheta}(y_j^{n,N}(t)-v_0)\right) \\ &\quad \times \exp\left(\sqrt{1-\rho^2}\int_0^t \sqrt{y_j^{n,N}}\,d\alpha_i^N\right). \end{aligned}$$

- Computation of *crude* quantized premium for N and M.
- Space Romberg log-extrapolation $\widehat{\mathrm{RCrAsCall}}^{Hest}(s_0, K)$.
- K-linear interpolation $\widehat{\mathrm{IRAsCall}}^{Hest}(s_0, K)$ based on the forward moneyness and the Call-Put parity formula

$$\mathrm{AsianCall}^{Hest}(s_0, K) - \mathrm{AsianPut}(s_0, K) = s_0\frac{1-e^{-rT}}{rT} - Ke^{-rT}.$$

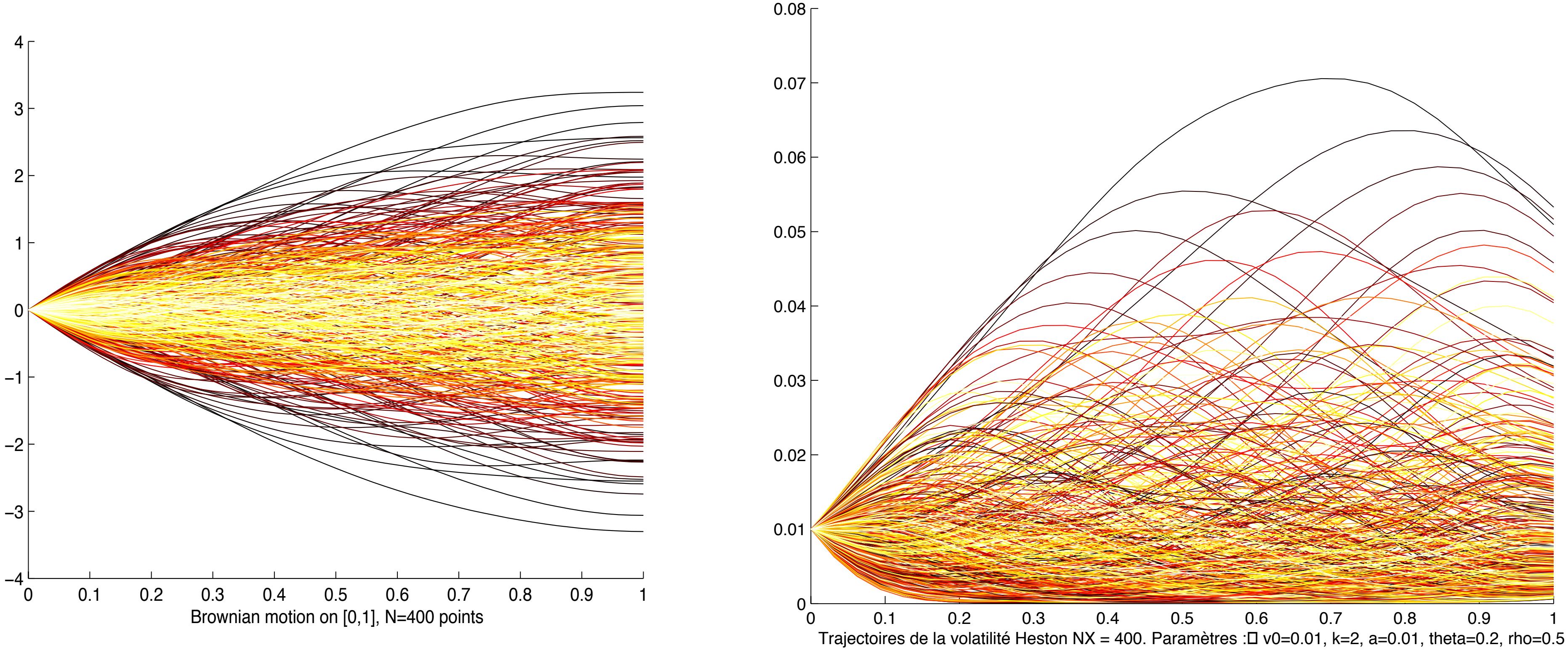

FIG. 4: Optimized Quantizer of the Heston volatility process $N = 400$

▷ Parameters of the Heston model :

$$\boxed{s_0 = 100,\ k = 2,\ a = 0.01,\ \rho = 0.5,\ v_0 = 10\%,\ \vartheta = 20\%.}$$

▷ Parameters of the option portfolio :

$$T = 1,\ K = 99, \cdots, 111 \qquad \text{(13 strikes)}.$$

▷ Reference price : computed by a 10^8 trial Monte Carlo simulation (including a time Romberg extrapolation with $2n = 256$, $n = 128$).

▷ Parameters of the quantization quadrature formulae :

$$\Delta t = 1/32, \qquad (N, M) = (400, 100),\ (1000, 100)\ \text{ or }\ (3200, 400)$$

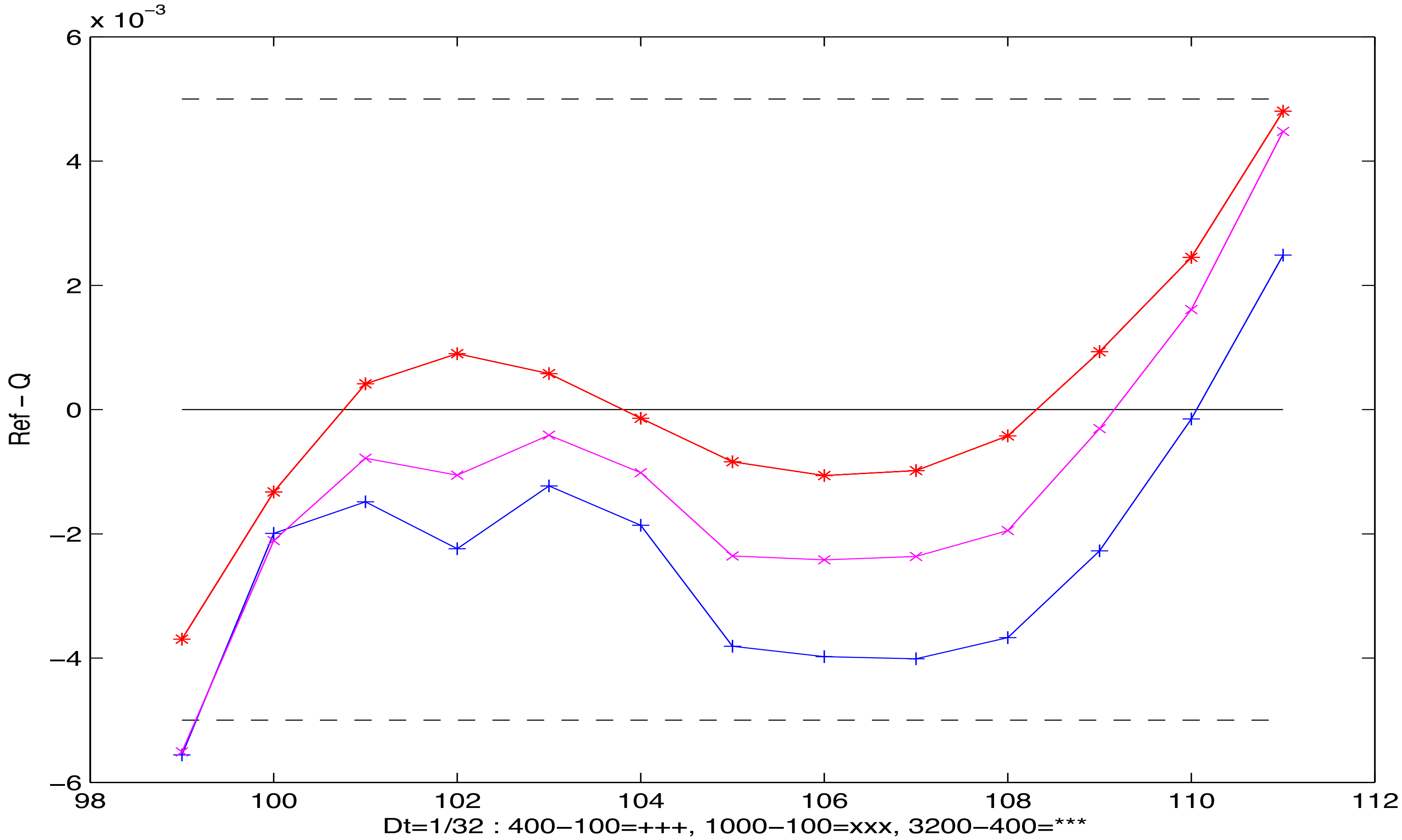

FIG. 5: K-Interpolated-log-Romberg extrapolated- FQ price : The error with $(N, M) = (400, 100)$, $(N, M) = (1000, 100)$, $(N, M) = (3200, 400)$

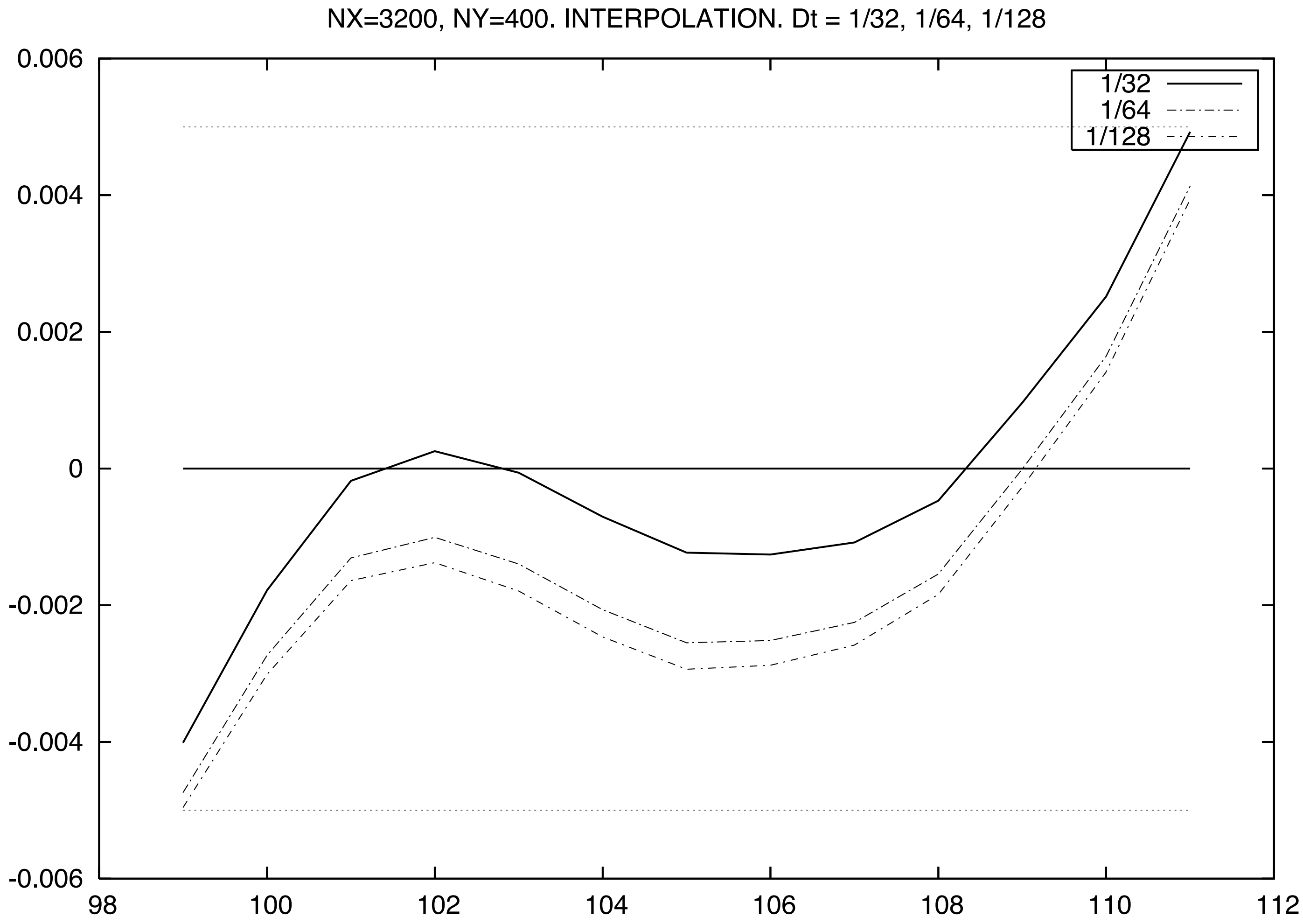

Fig. 6: K-Interpolated-log-Romberg extrapolated- FQ price : Convergence as $\Delta t \to 0$ with $(N, M) = (3200, 400)$

Conclusion

▷ Functional Quantization can compute a whole vector (more than 10) option premia for the Asian option in the Heston model

Within 1 cent accuracy in less than 1 second

(implementation in C on 2.5 GHz processor).

▷ Functional Quantization is not dedicated to the Heston model. Similar tests carried out in the B-S model, in progress with the SABR model.

▷ Perspective and projects : implementation of theoretical results for Lévy processes, other path-dependent options (barrier options, etc).

A possible alternative : The product quantization (G.P.-J. Printems, MCMA, 2006)

A (stationary) product quantization of the Brownian motion is defined by

$$\widehat{W}^N := \sum_{n \geq 1} \lambda_n \widehat{\xi}_n^{N_n} e_n^W, \qquad N_1 \times \cdots \times N_m \leq N$$

where $\widehat{\xi}_n^{N_n}$ are 1-dimensional (i.i.d.) optimal N_n-quantizations.

Less efficient (twice...) but all no storing constraint :

all the "ingredients" (scalar optimal quantizations, optimal size allocations, etc) can be computed from a ...

100×2 matrix !!

More about quantization on the new website

http ://quantification.finance-mathematique.com/

▷ Bibliography

▷ Download optimal/optimized Vector Quantizers of the normal distribution $\mathcal{N}(0; I_d)$, $1 \leq d \leq 10$.

▷ Download architectures of optimal product quantizers of the Brownian motion.

▷ Soon available : optimized quantizers of the Brownian motion.